A Dyck path with air pockets is a non-empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, and consisting of up-steps $U = (1, 1)$ and down-steps $D_k = (1, -k)$, $k \geq 1$, where two down steps cannot be consecutive (we set $D_0 = 0$, for short). The set of such paths is denoted by $A$.

Let $A_n$ denote the set of $n$-length Dyck paths with air pockets.

Example:

![Figure: A Dyck path with air pockets](image)

Cardinality formula (OEIS A004148):

$$|A_n| = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k} \binom{n-k}{k} \binom{k}{n-1-k}.$$  

Generating function:

$$\sum_{n=2}^\infty |A_n| \cdot x^n = \frac{1-x-x^2-\sqrt{1-2x+2x^2-2x^3+2x^4}}{2x}.$$  

A Dyck path with air pockets is prime if it ends with $D_k$, $k \geq 2$, and it returns to the x-axis only once. The set of such paths is denoted by $F$.

Lowered and elevated paths

We introduce two transformations of Dyck paths with air pockets. If $\alpha$ is a Dyck path with air pockets of the form $U^sD_t$ (where $s$ is either empty or in $A$), then we define the lowered version of $\alpha$ as $\alpha^\flat = U^s\beta D_t$. We also define the inverse operation $\hat{\flat}$, and call $\alpha^\flat$ the lowered version of $\alpha$.

Example:

![Figure: Lowered and elevated paths](image)

The operations $\flat$ and $\hat{\flat}$ will help us to define a bijection between $A_n$ and a class of well-known lattice paths.

Bijection with peakless Motzkin paths, $M_n$

A peakless Motzkin path is a non-empty lattice path in the first quadrant of $\mathbb{Z}^2$ starting at the origin, ending on the x-axis, consisting of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and flat-steps $F = (1, 0)$, having no occurrence of $UD$. The set of such paths is denoted by $M$. The set of $n$-length peakless Motzkin paths is denoted by $M_n$.

$$A \overset{\psi}{\longrightarrow} M$$

Theorem: The map $\psi$ induces a bijection between $A_n$ and $M_{n+1}$.

Pattern popularity in $A_n$ (2 ≤ $n$ ≤ 11) can be found in the following table:

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Pattern popularity in $A_n$</th>
<th>OEIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>1, 2, 5, 13, 32, 80, 204, 565, 1273, 3217</td>
<td>A193320</td>
</tr>
<tr>
<td>$D$</td>
<td>1, 0, 2, 3, 7, 17, 40, 97, 238, 587</td>
<td>A051291</td>
</tr>
<tr>
<td>$\Delta_k$</td>
<td>0, 1, 3, 6, 13, 30, 70, 167, 405</td>
<td>A201631(= $u_n$)</td>
</tr>
<tr>
<td>$\Delta_{k+}$</td>
<td>1, 0, 2, 3, 7, 17, 40, 97, 238, 587</td>
<td>A051291</td>
</tr>
<tr>
<td>$\Delta_{k-1}$</td>
<td>1, 1, 2, 5, 10, 24, 47, 137, 335, 825, 2025</td>
<td>A051291</td>
</tr>
<tr>
<td>$\Delta_{k+1}$</td>
<td>1, 1, 3, 5, 12, 27, 64, 154, 375, 922, etc.</td>
<td></td>
</tr>
</tbody>
</table>